GENERALIZED COMPLEX GEOMETRY AND THE POISSON SIGMA MODEL

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Abstract

The supersymmetric Poisson Sigma model is studied as a possible worldsheet realization of generalized complex geometry. Generalized complex structures alone do not guarantee non-manifest $N = (2,1)$ or $N = (2,2)$ supersymmetry, but a certain relation among the different Poisson structures is needed. Moreover, important relations of an additional almost complex structure are found, which have no immediate interpretation in terms of generalized complex structures.

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1 Introduction

Two-dimensional non-linear sigma models with extended supersymmetry [1, 2] recently attracted attention due to the relations to generalized complex geometry [3, 4]. On the one hand the complex structures of the model of ref. [2] can be mapped onto two twisted generalized complex structures [4–8]. On the other hand it is expected that generalized complex geometry plays an important role in the first order formulation of the sigma model[1]

$$\mathcal{L}_{FO} = \frac{1}{2} \int d^2\theta \left(i(D^\alpha X^i)A_{i\alpha} - \frac{1}{2}\mathcal{G}^{ij}(X)(A_j A_i) - \frac{1}{2}\mathcal{P}^{ij}(X)(A_j\gamma_* A_i) \right) , \quad (1.1)$$

with metric $\mathcal{G}^{ij}$ and anti-symmetric tensor $\mathcal{P}^{ij}$ as defining structures of the target space[2] [11,12]. Here the X^i are scalar superfields, the $A_{i\alpha}$ real spinorial superfields, i.e. their lowest components $\psi_{i\alpha}$ are Majorana spinors. The action (1.1) is manifestly invariant under global $N = (1,1)$ supersymmetry. As the $A_{i\alpha}$ live in T^* it is indeed natural to expect that additional non-manifest supersymmetries will require a map $\mathcal{J}$ from $T \oplus T^* \to T \oplus T^*$ with $\mathcal{J}^2 = -1$. Such a map is called generalized complex structure, if the natural indefinite metric on $T \oplus T^*$ is hermitian with respect to $\mathcal{J}$ and if the latter obeys an integrability condition with respect to the bracket

$$[X + \xi, Y + \eta]_C = [X, Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) \qquad (1.2)$$

for $\mathcal{X} = X + \xi \in T \oplus T^*$ [3, 4]. The latter can be written as

$$[\mathcal{X}, \mathcal{Y}]_C - [\mathcal{J}\mathcal{X}, \mathcal{J}\mathcal{Y}]_C + \mathcal{J}[\mathcal{J}\mathcal{X}, \mathcal{Y}]_C + \mathcal{J}[\mathcal{X}, \mathcal{J}\mathcal{Y}]_C = 0 \qquad (1.3)$$

In the present application it is convenient to write $\mathcal{J}$ in the form

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix} \qquad (1.4)$$

with $J : TM \to TM$, $P : T^*M \to TM$, $L : TM \to T^*M$ and $K : T^*M \to T^*M$ (M is now interpreted as the target space manifold of the action (1.1)). From $\mathcal{J}^2 = -1$ one obtains

$$J^i{}_j J^j{}_k + P^{ij}L_{jk} = -\delta^i{}_k , \qquad (1.5)$$

$$J^i{}_j P^{jk} + P^{ij}K_j{}^k = 0 , \qquad (1.6)$$

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[1] A brief overview of our conventions is given in appendix A. Further details of the notation are explained e.g. in [9,10].

[2] A Wess-Zumino type term could be added to this action as well, however we do not consider such models in the present work.

$$K_i{}^j K_j{}^k + L_{ij} P^{jk} = -\delta_i{}^k \,, \qquad\qquad (1.7)$$

$$K_i{}^j L_{jk} + L_{ij} J^j{}_k = 0 \,, \qquad\qquad (1.8)$$

while the hermiticity condition yields

$$J^i{}_j + K_j{}^i = 0 \,, \qquad\qquad P^{\{ij\}} = 0 \qquad\qquad L_{\{ij\}} = 0 \,. \qquad (1.9)$$

Finally the integrability condition can be written as four differential relations:

$$J^j{}_{[k} J^i{}_{l],j} + J^i{}_j J^j{}_{[k,l]} + P^{ij} L_{[jk,l]} = 0 \qquad\qquad (1.10)$$

$$P^{[i|l|} P^{jk]}{}_{,l} = 0 \qquad\qquad (1.11)$$

$$J^i{}_{l,j} P^{jk} + J^i{}_j P^{jk}{}_{,l} + J^k{}_{[l,j]} P^{ij} - J^j{}_l P^{ik}{}_{,j} = 0 \qquad\qquad (1.12)$$

$$J^j{}_i L_{[jk,l]} + J^j{}_{[k} L_{l]i,j} + J^j{}_k L_{jl,i} + L_{ij} J^j{}_{[l,k]} + L_{jk} J^j{}_{l,i} = 0 \qquad\qquad (1.13)$$

For the generic case in (1.1) the relation between extended supersymmetry and generalized complex geometry turned out to be very complicated (cf. [12]; notice that these authors start from $(1,0)$ supersymmetry that is different from the model considered here). In ref. [11] the problem has been solved by going partly on-shell together with the restriction that $\mathcal{G}$ must be invertible. Both works do not cover the Poisson Sigma model (PSM) [13–15], which has important applications in string theory due to the path integral interpretation [15] of Kontsevitch's $\star$-product [16]. In addition, many questions remained open in [11, 12] and a simple example as the PSM certainly can help to clarify the situation. Thus we consider in this work the action (1.1) in the limit of a PSM, where $\mathcal{G} = 0$ and $\mathcal{P}$ is Poisson. In particular, the relation between PSMs with generalized complex target space and PSMs with non-manifest extended supersymmetry is analyzed. It should be noted that supersymmetric extensions of the PSM with *manifest* $N = (4,4)$ supersymmetry have been obtained in [17, 18] using harmonic superspace techniques.

To perform the calculations it is preferable to write the action (1.1) in chiral components. With $\mathcal{E} = \mathcal{G} + \mathcal{P}$ this becomes

$$\mathcal{L}_{FO} = \frac{1}{2} \int \mathrm{d}^2\theta \left(i(D_- X^i) A_{i+} - i(D_+ X^i) A_{i-} + \mathcal{E}^{ij} A_{j+} A_{i-} \right) \,. \qquad (1.14)$$

The strategy to find non-manifest extended supersymmetry is now quite easy: One writes down the most general transformations for the component fields in (1.14). Then the invariance of the action under these transformations is checked, which will lead to a number of algebraic and differential relations. Finally the new supersymmetries must obey the supersymmetry algebra (A.2) and moreover commute with the manifest supersymmetry. The

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last point is automatically satisfied if the transformations are written in a superspace covariant form.

For an additional supersymmetry with transformation parameter ε^+, yielding a $N = (2,1)$ invariant theory, dimensional analysis tells us that

$$\delta_+ X^i = \varepsilon^+ D_+ X^j J^{+i}{}_j + \varepsilon^+ A_{j+} P^{+ij} , \tag{1.15}$$

$$\delta_+ A_{i+} = \sqrt{2}\varepsilon^+ \partial_{++} X^j L^+_{ij} - \varepsilon^+ D_+ A_{j+} K^{+j}_i + \varepsilon^+ D_+ X^j D_+ X^k M^+_{ijk}$$
$$+ \varepsilon^+ A_{j+} A_{k+} N^{+jk}_i + \varepsilon^+ D_+ X^j A_{k+} Q^{+k}_{ij} , \tag{1.16}$$

$$\delta_+ A_{i-} = \varepsilon^+ D_+ A_{j-} R^{+j}_i + \varepsilon^+ D_- A_{j+} S^{+j}_i + \varepsilon^+ A_{j+} A_{k-} Y^{+jk}_i + \varepsilon^+ A_{j+} D_- X^k U^{+j}_{ik}$$
$$+ \varepsilon^+ D_+ X^j A_{k-} V^{+k}_{ij} + \varepsilon^+ D_+ D_- X^j T^+_{ij} + \varepsilon^+ D_+ X^j D_- X^k W^+_{ijk} . \tag{1.17}$$

Analogous relations follow for an additional supersymmetry Q_-.

Before starting the calculations the relation to the more familiar second order formulation of the action (1.1) or (1.14) is outlined briefly. If $\mathcal{E}^{ij}$ is invertible the spinorial fields $A_{i\alpha}$ can be eliminated by the equations of motion from (1.14)

$$\frac{\delta}{\delta A_{i+}} \mathcal{S}_{FO} = -\frac{i}{2} D_- X^i + \frac{1}{2} \mathcal{E}^{ji} A_{j-} , \qquad \frac{\delta}{\delta A_{i-}} \mathcal{S}_{FO} = \frac{i}{2} D_+ X^i - \frac{1}{2} \mathcal{E}^{ij} A_{j+} . \tag{1.18}$$

Then the second order Lagrangian ($\mathcal{E}_{ij} = (\mathcal{E}^{-1})_{ij}$)

$$\mathcal{L}_{SO} = \frac{1}{2} \int d^2\theta \, D_+ X^i D_- X^j \mathcal{E}_{ji} = \frac{1}{2} \int d^2\theta \, D_+ X^i D_- X^j (g+b)_{ji} \tag{1.19}$$

with the corresponding symmetry transformations

$$\delta_+ X^I = \varepsilon^+ D_+ X^j I^{+i}{}_j \qquad I^+ = J^+ + iP^+ \mathcal{E}^{-1} \tag{1.20}$$

$$\delta_- X^I = \varepsilon^- D_- X^j I^{-i}{}_j \qquad I^- = J^- - i(\mathcal{E}^{-1} P^-)^T \tag{1.21}$$

is obtained. These transformations define two additional non-manifest supersymmetries, if the metric g is hermitian with respect to $I^\pm$ and if $I^\pm$ define covariantly constant complex structures [2]. The definition of $(g, H = db, I^+, I^-)$ is equivalent to a certain H-twisted generalized Kähler structure on M [4].

2   General Sigma Model

In a first step the invariance of the action and one relation of the supersymmetry algebra are analyzed. We exemplify the calculations at hand of the

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δ_+ transformation, the generalization to δ_- is straightforward. For simplicity, the superscript "+" for the various tensors in (1.15)-(1.17) is omitted in this section. Moreover, we do not yet impose any constraints on $\mathcal{E}^{ij}$. Most results of this section can be found in [12] already, they are reproduced here to clarify our notations and conventions. However, ref. [12] in many formulas assumes an invertible $\mathcal{E}$, a constraint that we do not impose here.

The invariance of the action (1.14) under (1.15)-(1.17) can now be studied order in order of the gauge potentials $A_{i\alpha}$. Going through all steps one finds eleven different conditions:

$$L_{\{ij\}} = -T_{\{ij\}} \tag{2.1}$$

$$M_{ijk} - W_{jki} = \frac{1}{2}L_{[ij,k]} - \frac{1}{2}T_{jk,i} \tag{2.2}$$

$$J^j{}_i + K_i{}^j = S_i{}^j - i\mathcal{E}^{kj}T_{ki} \tag{2.3}$$

$$K_{[k}{}^i{}_{,j]} + Q_{kj}{}^i - U_{jk}{}^i = i\big((\mathcal{E}^{li}T_{lj})_{,k} - \mathcal{E}^{li}W_{ljk}\big) \tag{2.4}$$

$$J^j{}_i + R_i{}^j = -i\mathcal{E}^{jk}L_{ki} \tag{2.5}$$

$$V_{[ij]}{}^k = i\big(\mathcal{E}^{kl}M_{l[ij]} - (\mathcal{E}^{kl}L_{l[i})_{,j]}\big) \tag{2.6}$$

$$\mathcal{E}^{ij}N_j{}^{kl} = \frac{1}{2}\big(\mathcal{E}^{j[k}Y_j{}^{l]i} - \mathcal{E}^{i[l}{}_{,j}P^{j|k]}\big) \tag{2.7}$$

$$P^{\{ij\}} = i\mathcal{E}^{k\{i}S_k{}^{j\}} \tag{2.8}$$

$$P^{ij} = -i(\mathcal{E}^{ik}K_k{}^j + \mathcal{E}^{kj}R_k{}^i) \tag{2.9}$$

$$2N_i{}^{jk} = i\mathcal{E}^{l[j}U_{li}^{k]} - \frac{1}{2}P^{[kj]}{}_{,i} + \frac{i}{2}(\mathcal{E}^{l[j}S_l{}^{k]})_{,i} \tag{2.10}$$

$$Y_i{}^{jk} = i\big(\mathcal{E}^{lj}V_{li}{}^k + (\mathcal{E}^{kl}K_l{}^j)_{,i} + \mathcal{E}^{kl}Q_{li}{}^j + \mathcal{E}^{kj}{}_l J^l{}_i)\big) \tag{2.11}$$

In particular we obtain in the limit of $\mathcal{E} \to 0$

$$R_i{}^j = -J^j{}_i \ , \qquad\qquad P^{ij} = V_{[ij]}{}^k = N_i{}^{jk} = Y_i{}^{jk} = 0 \ . \tag{2.12}$$

More involved is the calculation of the supersymmetry algebra. The definition $\delta_+\Psi = i[\varepsilon^+Q_+, \Psi]$ implies that

$$\delta_+^1\delta_+^2\Psi = \sqrt{2}\varepsilon_1^+\varepsilon_2^+\partial_{++}\Psi \ . \tag{2.13}$$

We start with the commutator acting on X^i. First of all the fact that the derivative term on the rhs of (2.13) is generated yields as condition exactly eq. (1.5). Therefore, three out of four components of the generalized complex structure are already identified. Consequently, we also *choose* $L_{\{ij\}} = P^{\{ij\}} = 0$, else we do not obtain the desired structure[3]. Next we consider terms

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[3]The symmetric part of L and T can be set to zero by means of a "field equation"

$\propto D_+X^iD_+X^j$. If

$$M_{ijk} = \frac{1}{2}L_{[ij,k]} \tag{2.14}$$

these terms are equivalent to (1.10). The terms $\propto D_+A_{i+}$ lead to (1.6) and this identifies the last map of $\mathcal{J}$. The remaining conditions can be identified with (1.11) and (1.12) if

$$Q_{ij}{}^k = J^k{}_{[i,j]} \; , \qquad\qquad N_i{}^{kl} = \frac{1}{2}P^{kl}{}_{,i} \; . \tag{2.15}$$

Thus the condition that $\delta_+^1\delta_+^2 X^i = \sqrt{2}\varepsilon_1^+\varepsilon_2^+\partial_{++}X^i$, under the assumption that the target-space is generalized complex, constrains the transformation of A_{i+} as

$$\begin{aligned}
\delta_+A_{i+} = &\sqrt{2}\varepsilon^+\partial_{++}X^jL_{ij} + \varepsilon^+D_+A_{j+}J^j{}_i + \frac{1}{2}\varepsilon^+D_+X^jD_+X^kL_{[ij,k]} \\
&+ \frac{1}{2}\varepsilon^+A_{j+}A_{k+}P^{jk}{}_{,i} + \varepsilon^+D_+X^jA_{k+}J^k{}_{[i,j]} \; .
\end{aligned} \tag{2.16}$$

Up to conventions this is exactly the result of [12]. As $\delta_+^1\delta_+^2 A_{i+}$ is obviously independent of A_{i-} we can borrow the result therefrom that this transformation satisfies the supersymmetry algebra relation if the target space is generalized complex.

Now we should go again through the conditions (2.1)-(2.11). From (2.1)-(2.4), (2.6) and (2.11) follows

$$T_{\{ij\}} = 0 \; , \qquad W_{ijk} = \frac{1}{2}T_{ij,k} \; , \qquad S_i{}^j = i\mathcal{E}^{kj}T_{ki} \; , \tag{2.17}$$

$$U_{ij}{}^k = -\frac{i}{2}\mathcal{E}^{lk}T_{li,j} - i\mathcal{E}^{lk}{}_{,j}T_{li} \; , \qquad V_{[ij]}{}^k = i\mathcal{E}^{kl}L_{ij,l} - i\mathcal{E}^{kl}{}_{,[i}L_{j]l} \; , \tag{2.18}$$

$$Y_i{}^{jk} = i(\mathcal{E}^{lj}V_{li}{}^k - \mathcal{E}^{kl}J^j{}_{i,l} - \mathcal{E}^{kl}{}_{,i}J^j{}_l + \mathcal{E}^{kj}{}_{,l}J^l{}_i) \; . \tag{2.19}$$

Furthermore, the relations (2.8) and (2.10) are automatically satisfied.

At this point we should have a careful look at the definition of P in (2.9). Together with (2.5) and the split $\mathcal{E} = \mathcal{G} + \mathcal{P}$ one finds after some algebra that

$$P^{ij} = -i(\mathcal{P}K)^{[ij]} - (\mathcal{P}L\mathcal{P})^{ij} - (\mathcal{G}L\mathcal{G})^{ij} \tag{2.20}$$

is the correct definition of the anti-symmetric tensor P. However, the definition (2.9) contains a symmetric part as well, which must vanish. This leads to the constraint

$$i(\mathcal{G}K)^{\{ij\}} + (\mathcal{P}L\mathcal{G})^{ij} + (\mathcal{G}L\mathcal{P})^{ij} = 0 \; . \tag{2.21}$$

symmetry and an appropriate redefinition of various quantities in (1.16) and (1.17). In general, this is not possible for the symmetric part of P, as the corresponding term $\propto PA_+$ in (1.15) is not part of a field equation.

It is seen that the constraint is trivial for $\mathcal{G} = 0$, on the other hand it reduces for $\mathcal{P} = 0$ to the condition that the metric $\mathcal{G}$ is hermitian with respect to K.

As a side remark we mention that under the elimination of $A_{i\alpha}$ in eq. (1.18) $I^+ = \mathcal{E}(K^+\mathcal{E}^{-1} - iL^+)$ automatically squares to -1 due to the constraints derived in this section (cf. also [12]). If $\mathcal{P} = 0$ it is easily seen that $[\mathcal{J}^+, \mathcal{J}^-] = 0$ implies $[I^+, I^-] = 0$, i.e. in this special case there exists a simple relation between the generalized complex structures of the first order model and of the generalized Kähler structure resp.

3   Supersymmetric PSM

In the current work we want to investigate the case where the action (1.1) reduces to a supersymmetric PSM. This allows an important simplification of the analysis: as the Poisson tensor can be transformed to Casimir-Darboux coordinates it is sufficient to consider a constant $\mathcal{P}$ for any analysis local in the target space manifold. Thus we consider in the following the case where $\mathcal{E}$ reduces to a constant and anti-symmetric matrix, denoted by $\mathcal{P}$.

As is obvious from (1.9) and (1.12) P is a Poisson tensor. It is crucial to distinguish this tensor clearly from $\mathcal{P}$ in (1.1). In the current situation, $\mathcal{P}$ is Poisson as well, but the two Poisson structures are not equivalent, though they are related. In particular it follows, that the rank of P cannot be larger than the rank of $\mathcal{P}$, but P need not be constant for constant $\mathcal{P}$. Thus the relations (1.5), (1.6) and (1.10)-(1.12) should be investigated with the restriction $P^{ij} = -i(\mathcal{P}K)^{[ij]} - (\mathcal{P}L\mathcal{P})^{ij}$. Eq. (1.5) is most elegantly written as

$$(R^2)_i{}^j = -\delta_i{}^j \, , \tag{3.1}$$

i.e. R is an almost complex structure. Then with the use of (1.8) eq. (1.6) is satisfied identically. Eqs. (1.10)-(1.12) yield complicated differential conditions that we do not reproduce in full generality here. Notice that the relation (2.7), the only restriction from the invariance of the action that we did not solve in the previous section, is satisfied identically in the case of the PSM.

Now, the derivation of the remaining commutators splits into to parts: First we consider the extension to $N = (2,1)$ supersymmetry. There the only remaining commutator is $[\delta_+^1, \delta_+^2]A_{i-}$. Then one has to ensure that the commutators $[\delta_+, \delta_-]\Psi$ vanish for all fields Ψ.

3.1 $\mathbf{N} = (\mathbf{1}, \mathbf{1}) \to \mathbf{N} = (\mathbf{2}, \mathbf{1})$

While the the commutators $[\delta_+^1, \delta_+^2]\Psi$ could be solved for X^i and A_{i+} off-shell in full generality, unraveling the structure of a generalized complex geometry at the target space, this does not seem to be possible for the remaining field A_{i-} (cf. the complicated relations in [12], esp. eq. (A.3)). Indeed, it is not expected that the non-manifest supersymmetry closes off-shell, except for certain very special cases. Therefore, only on-shell closure of the algebra will be demanded in the following and consequently the transformations (1.15)-(1.17) together with the restrictions derived in the previous section can be reduced to

$$\delta_+ X^i = -i\varepsilon^+ A_{j+}(\mathcal{P}R)^{ij} \ , \tag{3.2}$$

$$\delta_+ A_{i+} = -\varepsilon^+ D_+ A_{j+} R_i{}^j - i\varepsilon_+ A_{j+} A_{k+} \mathcal{P}^{jl} R_i{}^k{}_{,l} \ , \tag{3.3}$$

$$\delta_+ A_{i-} = \varepsilon^+ D_+ A_{j-} R_i{}^j - i\varepsilon_+ A_{j-} A_{k+} \mathcal{P}^{jl} R_i{}^k{}_{,l} \ . \tag{3.4}$$

Not surprisingly, the symmetry transformations now can be written in terms of a single almost complex structure, which is in agreement with previous results [2,11]. Nevertheless, notice the difference to these approaches. As we do not insist on $\mathcal{P}$ having full rank, the equations of motion cannot be solved for $A_{i\alpha}$ (cf. (1.18)) and the transformation rule for X^i cannot be reduced to the form (1.20). Of course the relation (3.2) looks rather strange, in particular one obtains $\delta X^i = 0$ for BF theory. But in that case $D_\alpha X^i$ already is an equation of motion and therefore the representation of supersymmetry on that field vanishes on-shell.

Now, the derivation of $[\delta_+^1, \delta_+^2]A_{i-}$ is surprisingly easy. As R is almost complex, the correct supersymmetry algebra is generated in an obvious way. All remaining contribution are found to vanish if the modified integrability condition

$$\mathcal{P}^{m[k} R_i{}^{|j|} R_j{}^{l]}{}_{,m} - \mathcal{P}^{mj} R_j{}^{[k} R_i{}^{l]}{}_{,m} = 0 \tag{3.5}$$

holds. First notice that this relation is satisfied for any Poisson tensor $\mathcal{P}$ if $R \equiv K$, i.e. $L \equiv 0$. A simple interpretation can be given if $\mathcal{P}$ can be used as an intertwiner that defines a new map $TM \to TM$ as[4]

$$\mathcal{P}R = \tilde{I}\mathcal{P} \ . \tag{3.6}$$

In the case of a symplectic $\mathcal{P}$ the new almost complex structure $\tilde{I}$ is exactly I of eq. (1.20). Then the differential condition (3.5) is nothing but the integrability condition (vanishing Nijenhuis tensor) of I already found in [2].

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[4]A similar observation has been made in [12] as well, but these authors used a different intertwiner.

If $\mathcal{P}$ does not have full rank, I does no longer exist, but on each symplectic leaf a similar structure still can be defined. By choosing $\tilde{I}$ to be this almost complex strucutre, (3.5) reduces the integrability condition for $\tilde{I}$ if R is of the form $R = R_0 \oplus R_1$, where R_0 lives in the symplectic leaf, while $\mathcal{P}R_1 \equiv 0$. Furthermore, it can be checked that (3.5) is identically zero for all components of R_1. It should be stressed that this is not the most general solution of (3.5) as R needs not be decomposable in this way.

The tensors T_{ij} and $V_{ij}{}^k$ remain undetermined. As all components of $\delta_+ A_{i+}$ and $\delta_+ X^i$ have been fixed by the requirement that the target-space is generalized complex this ambiguity is not related to "field equation" symmetries. Of course, the off-shell closure of some transformations is somehow arbitrary, if this constraint is not imposed for all fields. However, this is sufficient to analyze under which conditions the sigma model, whose target space is equipped with a generalized complex structure, has an additional non-manifest supersymmetry.

For completeness the conditions for off-shell closure are reproduced for the special case of BF theory. In this simple model $P \equiv 0$, $R \equiv K$ and the correct supersymmetry algebra is ensured by the fact that $J^2 = -1$. All other contributions have to vanish. The relations involving T can be cast into the form

$$T_{kj}J^j{}_i + T_{ij}J^j{}_k = 0 \ , \tag{3.7}$$

$$T_{ij}J^j{}_{k,l} - T_{kj}J^j{}_{i,l} = 0 \ , \qquad (T_{kj}J^j{}_{[i})_{,l]} = T_{[i|k,j}J^j{}_{l]} \ , \tag{3.8}$$

$$V_{\{il\}}{}^j T_{jk} = J^j{}_{\{i}T_{l\}k,j} + (T_{jk}J^j{}_{\{i})_{,l\}} - \frac{1}{2}J^j{}_{\{i|,k}T_{j|l\}} \ , \tag{3.9}$$

$$(J^j{}_{[k,l}T_{ji})_{,m} + (J^j{}_{[k}T_{l]i,j})_{,m} + J^j{}_i T_{j[k,l]m} + J^j{}_{[k,l]m}T_{ji} = V_{i[k}{}^j T_{l]j,m} \ . \tag{3.10}$$

Obviously, $T \equiv 0$ is a simple and appealing solution for this case. The other immediate guess $T \propto L$ is not possible in general, as the above set of differential conditions does not reduce to (1.5)-(1.13).

The remaining conditions yield algebraic and differential equations for V:

$$J^j{}_{[i}V_{k]j}{}^l = 0 \tag{3.11}$$

$$J^k{}_{j,l}J^j{}_i + J^k{}_{i,j}J^j{}_l = J^k{}_j V_{il}{}^j - J^j{}_i V_{jl}{}^k \tag{3.12}$$

$$-J^j{}_i V_{j[k}{}^m{}_{,l]} + J^j{}_{[k,l]}V_{ij}{}^m + J^j{}_{[k}V_{l]i}{}^m{}_{,j} = V_{i[k}{}^j V_{l]j}{}^m \tag{3.13}$$

Notice the similarity between (3.11) and the condition (1.10) for $P = 0$: Indeed, with (2.12) and (2.15) we can write the latter as $J^j{}_{[i}Q_{k]j}l = 0$. However, $Q_{ij}{}^k$ is antisymmetric in its lower indices while $V_{ij}{}^k$ is symmetric. Due to this characteristic, the rhs of (3.12) is found to vanish when antisymmetrized in i and l. But this does not lead to a new constraint for J, as the lhs is found to reduce to (1.10) in that case.

One might be tended to choose $V_{ij}{}^k = 0$ as well. But this yields an additional constraint onto the generalized complex structure, as the integrability condition (1.10) must split into two independent pieces according to (3.12).

We do not go into further details about the off-shell closure of the PSM. Notice however, that $T \equiv 0$ will again be a solution of the $N = (2,1)$ extension. On the other hand, the differential conditions for V become much more involved than in the case above. Unfortunately, it does not seem to be possible to bring them into a form similar to (3.11)-(3.13) by substituting $J \to -R^T$, as one might expect naively.

3.2 N = (1, 1) → N = (2, 2)

To implement $N = (2,2)$ supersymmetry the result found so far for δ_+ must be generalized to δ_-. But this is almost trivial: first one has to exchange all indices $+ \leftrightarrow -$, moreover the indices of all tensors $\mathcal{E}$ have to be exchanged: $\mathcal{E}^{ij} \to \mathcal{E}^{ji}$. For the special case of the PSM this boils down to add a sign in front of every $\mathcal{P}^{ij}$.

Now the tensors from δ_+ and δ_- must be distinguished again by the use of the labels according to (1.15)-(1.17). Then the two supersymmetries are found to commute if the two conditions

$$[R^+, R^-] = 0 \ , \tag{3.14}$$

$$\mathcal{P}^{km} R_i^{+j} R_j^{-l}{}_{,m} - \mathcal{P}^{lm} R_i^{-j} R_j^{+k}{}_{,m} + (\mathcal{P}R^+)^{mk} R_i^{-l}{}_{,m} - (\mathcal{P}R^-)^{ml} R_i^{+k}{}_{,m} = 0 \tag{3.15}$$

hold. Notice that (3.15) reduces to (3.5) for $R^+ = R^-$. Furthermore, for symplectic $\mathcal{P}$ these conditions should reduce to the ones found in [2] (cf. also [11]). It is interesting to study the relation of (3.14) to a possible constraint $[\mathcal{J}^+, \mathcal{J}^-] = 0$. With the definition of R in eq. (2.5) the commutator (3.14) becomes (recall the different sign in the definition of R^-)

$$[K^+, K^-] + [L^+\mathcal{P}, L^-\mathcal{P}] + i[K^+, L^-\mathcal{P}] - i[L^+\mathcal{P}, K^-] = 0 \ , \tag{3.16}$$

while the relevant commutator from $[\mathcal{J}^+, \mathcal{J}^-] = 0$ is found to be

$$[K^+, K^-] - [L^+\mathcal{P}, L^-\mathcal{P}] + i(L^+\mathcal{P}K^- - L^-J^-\mathcal{P} + L^-\mathcal{P}K^+ - L^-J^+\mathcal{P}) = 0 \ . \tag{3.17}$$

The difference of these two equations does not vanish by means of the remaining conditions form $[\mathcal{J}^+, \mathcal{J}^-] = 0$. Therefore we conclude that the constraints (3.14) and (3.15) cannot be interpreted in straightforward way as parts of the generalized complex structure.

It may be useful to summarize at this point the conditions derived for an $N = (2,2)$ supersymmetric PSM: It was assumed that $J^\pm$, $P^\pm$, $L^\pm$ and $K^\pm$ in (1.15) and (1.16) define two generalized complex structures. Then the transformations of $A_{i\pm}$ under $\delta_\pm$ are given by (2.16). The remaining transformations $\delta_\pm A_{i\mp}$ have been studied for on-shell closure, only. These transformations depend on two almost complex structures $R^\pm$ that must satisfy (3.5), (3.14) and (3.15).

4   Conclusions

In this work the relation between extended non-manifest supersymmetry of the Poisson Sigma model and generalized complex structures has been studied. As expected, not every PSM with extended supersymmetry constrains its target space to be generalized complex. The converse is true neither: beside the relations among the Poisson structures of generalized complex geometry and of the PSM resp., it was found that an additional almost complex structure must obey a (modified) integrability condition, which in certain cases can be interpreted as a complex structure restricted to the symplectic leaves of the Poisson manifold. Finally the conditions for $N = (2,2)$ supersymmetry have been analyzed, the ensuing conditions do not necessarily imply that the two generalized complex structures commute.

Among the unanswered question there remains the interpretation of certain differential conditions, esp. the constraint of vanishing Nijenhuis tensor of the Poisson structure P in (1.11). Also, the conditions for off-shell closure of the algebra could not be solved. Here important additional constraints on the target space appear that we were not yet able to interpret in a conclusive way.

Of course, it would be interesting to study extensions of the model, e.g. the inclusion of a Wess-Zumino term [19], where a twisted generalized complex structure is expected [12], or non-topological extensions. Finally, we did not consider global effects, such as changes of the rank of the Poisson tensor $\mathcal{P}$ or effects from non-trivial boundary conditions.

Acknowledgement

(FWF).

A Notations and Conventions

The conventions used are explained in detail in [9, 10]. As they differ from the ones in [11, 12] the most important definitions are summarized in this appendix.

The γ-matrices are used in a chiral representation:

$$\gamma^0{}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \gamma^1{}_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \gamma_{*\alpha}{}^\beta = (\gamma^1\gamma^0)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.1}$$

As we work with spinors in a chiral representation we can use $\chi^\alpha = (\chi^+, \chi^-)$ and upper and lower chiral components are related by $\chi^+ = \chi_-$, $\chi^- = -\chi_+$, $\chi^2 = \chi^\alpha \chi_\alpha = 2\chi_- \chi_+$. Furthermore for Majorana spinors χ^+ is real while χ^- is imaginary. Vectors in light-cone coordinates coincide with the respective spin tensor decomposition if they are defined as $v^{++} = \frac{i}{\sqrt{2}}(v^0 + v^1)$, $v^{--} = \frac{-i}{\sqrt{2}}(v^0 - v^1)$.

Finally the basic conventions of $(1,1)$ supersymmetry are explained. The representation of the supercharges is chosen as

$$Q_\alpha = \partial_\alpha - i(\gamma^a\theta)_\alpha \partial_a \ , \qquad \{Q_\alpha, Q_\beta\} = 2i\gamma^a_{\alpha\beta}\partial_a \ , \tag{A.2}$$

which yields as a convenient choice of the supersymmetry-covariant derivatives $D_\alpha = \partial_\alpha + i(\gamma^a\theta)_\alpha \partial_a$. In chiral components these derivatives obey

$$\{D_+, D_-\} = 0 \ , \qquad D_+^2 = -\sqrt{2}\partial_{++} \ , \qquad D_-^2 = -\sqrt{2}\partial_{--} \ . \tag{A.3}$$

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